REILLY TYPE INEQUALITY FOR THE FIRST EIGENVALUE OF THE $L_{r;F}$ OPERATOR

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Abstract. Given a positive function F on $\mathbb{S}^n$ which satisfies a convexity condition, for $1 \le r \le n$, we define for hypersurfaces in $\mathbb{R}^{n+1}$ the r-th anisotropic mean curvature function $H_{r;F}$, a generalization of the usual r-th mean curvature function. We also define $L_{r;F}$ operator, the linearized operator of the r-th anisotropic mean curvature, which is a generalization of the usual L_r operator for hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. The Reilly type inequalities for the first eigenvalue of the $L_{r;F}$ operator have been proved.

1. Introduction

A classical result of Reilly [17] establishes that the first positive eigenvalue λ_1 of the Laplacian operator Δ of a closed (that is, compact and without boundary) hypersurface M immersed into the Euclidean space $\mathbb{R}^{n+1}$ satisfies

$$\lambda_1 \le \frac{n}{\mathrm{vol}(M)} \int_M H^2 dM,$$

where H denotes the mean curvature of M, with equality if and only if M is a round sphere in $\mathbb{R}^{n+1}$. More generally, Reilly obtained that

$$\lambda_1 \left(\int_M H_r dM \right)^2 \le n \,\mathrm{vol}(M) \int_M H_{r+1}^2 dM,$$

for every $0 \le r \le n-1$, where H_r stands for the r-th mean curvature of the hypersurface, and equality holds precisely when M is a round sphere (recall that $H_0 = 1$ by definition, and $H_1 = H$).

The first eigenvalue of L_r, the linearized operator of the r-th mean curvature of hypersurfaces in $\mathbb{R}^{n+1}$, has been studied by do Carmo, Alencar and Rosenberg

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[2], they proved, under the hypothesis $H_{r+1} > 0$, that

$$\lambda_1^{L_r} \int_M H_r \leq (n-r)C_n^r \int_M H_{r+1}^2,$$

and equality holds precisely if M is a sphere of $\mathbb{R}^{n+1}$.

More recently, Veeravalli [20] has extended Reilly's inequalities to the case of hypersurfaces immersed into hyperbolic and spherical spaces. See also [7], [8] and [11] for other extensions of Reilly's inequalities.

In [3], L. J. Alías and J. M. Malacarne proved:

Theorem 1.1. *Let $\psi : M \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface immersed into the Euclidean space. Assume that L_r is elliptic on M, for some $0 \leq r \leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then, for every $0 \leq s \leq n-1$ it follows that*

$$\lambda_1^{L_r}\left(\int_M H_s dM\right)^2 \leq c(r) \int_M H_r dM \int_M H_{s+1;F}^2 dV, \ c(r) = (n-r)C_n^r. \tag{1.1}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

In this paper, we prove an anisotropic version of Theorem 1.1. First we introduce some notations.

Let

$$\mathbb{S}^n = \{y \in \mathbb{R}^{n+1} \mid \|y\| = 1\}$$

be the standard unit sphere in the Euclidean space $\mathbb{R}^{n+1}$, where $\|y\|^2 = \sum_{i=1}^{n+1}(y^i)^2$ for $y = (y^1, y^2, \cdots, y^{n+1}) \in \mathbb{R}^{n+1}$.

Let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$(D^2F + FI)_y > 0, \quad \forall y \in \mathbb{S}^n, \tag{1.2}$$

where D^2F denotes the intrinsic Hessian of F on $\mathbb{S}^n$ and I denotes the identity on $T_y\mathbb{S}^n$, > 0 means that the matrix is positive definite.

We consider the map

$$\begin{aligned} \phi \colon & \mathbb{S}^n \to \mathbb{R}^{n+1}, \\ & y \to F(y)y + (\operatorname{grad}_{\mathbb{S}^n} F)_y, \end{aligned} \tag{1.3}$$

its image $W_F = \phi(\mathbb{S}^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [5], [12], [14], [19], [21]). When $F \equiv 1$, the Wulff shape W_F is just $\mathbb{S}^n$.

Now let $x : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact, oriented hypersurface without boundary. Let $N : M \to \mathbb{S}^n$ denote its Gauss map. The map $\nu = \phi \circ N : M \to W_F$ is called the anisotropic Gauss map of x.

Let $S_F = -\mathrm{d}\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\kappa_1, \kappa_2, \cdots, \kappa_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r} \quad (1 \leq r \leq n).$$

We set $\sigma_0 = 1$. The r-th anisotropic mean curvature $H_{r;F}$ is defined by $H_{r;F} = \sigma_r / C_n^r$, also see Reilly [15]. $H_F = H_{1;F}$ is called the anisotropic mean curvature. When $F \equiv 1$, $H_{r;F}$ is just the r-th mean curvature H_r of hypersurfaces which has been studied by many authors (see [1], [2], [3], [13]). Thus, the r-th anisotropic mean curvature $H_{r;F}$ generalizes the r-th mean curvature H_r of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

Associated to each $H_{r;F}$, we define its linearized operator $L_{r;F}$ (see Section 3), denotes $\lambda_1^{L_{r;F}}$ its first positive eigenvalue.

In section 2, we define a constant μ depends on the function $F : \mathbb{S}^n \to \mathbb{R}^+$.

In this paper, we prove the following Reilly type inequality:

Theorem 1.2. *Let $X \colon M \to \mathbb{R}^{n+1}$ be a compact oriented hypersurface without boundary immersed into Euclidean space, and let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth function which satisfies the convexity condition (1.2). If $H_{r;F} > 0$ for some $r = 1, \cdots, n$. Then the first eigenvalue $\lambda_1^{L_{r;F}}$ of $L_{r;F}$ satisfies*

$$\lambda_1^{L_{r;F}} (\int_M H_{s-1;F} dV)^2 \leq \mu c(r) \int_M H_{r;F} dV \int_M H_{s;F}^2 dV, \quad s = 1, 2, \cdots, n, \tag{1.4}$$

where $c(r) = (n-r)C_n^r$.

2. Preliminaries

We define $F^* : \mathbb{R}^{n+1} \to \mathbb{R}$ to be:

$$F^*(y) = \sup \left\{ \frac{\langle y, z \rangle}{F(z)} | \ z \in \mathbb{R}^{n+1} \setminus \{0\} \right\}, \tag{2.1}$$

then F^* is a Minkowski norm on $\mathbb{R}^{n+1}$. In fact, as proved in [10], $F^* : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is smooth and we have

Proposition 2.1. (1) *$F^*(y) > 0$, $\forall y \in \mathbb{R}^{n+1} \setminus \{0\}$;*
(2) *$F^*(ty) = tF^*(y)$, $\forall y \in \mathbb{R}^{n+1}, t > 0$;*

(3) $F^*(y+z) \le F^*(y) + F^*(z)$, $\forall y, z \in \mathbb{R}^{n+1}$, *and the equality holds if and only if* $y = 0$, *or* $z = 0$ *or* $y = kz$ *for some* $k > 0$.
(4) $W_F = \{y \in \mathbb{R}^{n+1} |\ F^*(y) = 1\}$.

We define

$$\bar{g}_{\alpha\beta}(y) = \frac{1}{2}\frac{\partial^2 (F^*)^2}{\partial y^\alpha \partial y^\beta}(y), \tag{2.2}$$

and

$$g_y(X, Y) = \bar{g}_{\alpha\beta}(y) X^\alpha Y^\beta, \tag{2.3}$$

where $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $X = (X^1, X^2, \cdots, X^{n+1})$, $Y = (Y^1, Y^2, \cdots, Y^{n+1}) \in T_y\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

For $u, v \in \mathbb{R}^{n+1}$, $y \in \mathbb{R}^{n+1} \setminus \{0\}$, we write

$$\langle u, v\rangle_y = g_y(u, v), \quad \|u\|_y = \sqrt{g_y(u, u)},$$

where $u = (u_1, \cdots, u_{n+1})$, $v = (v_1, \cdots, v_{n+1})$.

Define constant λ, Λ, μ by

$$\lambda = \lambda(\mathbb{R}^{n+1}, F) = \inf_{y,u\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\|u\|_y^2}{\|u\|^2},$$

$$\Lambda = \Lambda(\mathbb{R}^{n+1}, F) = \sup_{y,u\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\|u\|_y^2}{\|u\|^2}.$$

$$\mu = \mu(\mathbb{R}^{n+1}, F) = \frac{\Lambda}{\lambda}.$$

It is clear that $1 \le \mu < \infty$, and $\mu = 1$ if and only if F^* is an Euclidean norm.

Let $x : M \to \mathbb{R}^{n+1}$ be a compact oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let $\nu : M \to W_F$ denote its anisotropic Gauss map. Then for any $p \in M$, $\nu(p)$ is perpendicular to T_pM with respect to the inner product $g_{\nu(p)}$ and $F^*(\nu(p)) = 1$. Thus, we call $\nu(p)$ an anisotropic unit normal vector of T_pM.

Let $\overline{\nabla}$ be the standard connection on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For vector fields X, Y on M, we decompose $\overline{\nabla}_X Y$ as the tangent part $\nabla_X Y$ and the anisotropic normal part $\mathrm{II}(X, Y)\nu$ with respect to the inner product g_ν. That is:

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)\nu, \tag{2.4}$$

where $g_\nu(\nabla_X Y, \nu) = 0$.

It is easy to verify that ∇ is a torsion free connection on M and II is a symmetric second order covariant tensor field on M. We call II the anisotropic second fundamental form.

Let $\{e_i\}_{i=1}^n$ be a local frame of M and $\{\omega^i\}_{i=1}^n$ its dual frame. Let $g_{ij} = g_\nu(e_i, e_j)$, $\nabla e_i = \omega_i^j \otimes e_j$, $\mathrm{II}(e_i, e_j) = h_{ij}$, $h_i^j = g^{jk}h_{ki}$, where (g^{ij}) is the inverse matrix of (g_{ij}). Then we have

$$dx = \omega^i e_i, \tag{2.5}$$

$$de_i = \omega_i^j e_j + h_{ij}\omega^j \nu, \tag{2.6}$$

$$d\nu = -h_i^j \omega^i e_j. \tag{2.7}$$

Differentiate (2.5) and using (2.6), we get

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{2.8}$$

$$h_{ij} = h_{ji}. \tag{2.9}$$

Differentiate (2.6) and using (2.6-2.7), we get

$$h_{ijk} = h_{ikj}, \tag{2.10}$$

$$d\omega_i^j - \omega_i^k \wedge \omega_k^j = -\frac{1}{2} R_i{}^j{}_{kl} \omega^k \wedge \omega^l, \tag{2.11}$$

where

$$h_{ijk}\omega^k = dh_{ij} - h_{ik}\omega_j^k - h_{kj}\omega_i^k,$$
$$R_i{}^j{}_{kl} = -R_i{}^j{}_{lk} = h_{ik}h_l^j - h_{il}h_k^j.$$

Differentiate (2.7) and using (2.6), we get

$$h_i{}^j{}_k = h_k{}^j{}_i, \tag{2.12}$$

where

$$h_i{}^j{}_k \omega^k = dh_i^j + h_i^k \omega_k^j - h_k^j \omega_i^k.$$

Note (h_i^j) is the matrix of the F-Weingarten operator $S_F = -d\nu$, its eigenvalues are called the anisotropic principal curvatures, we denote them by $\kappa_1, \cdots, \kappa_n$. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\kappa_1, \kappa_2, \cdots, \kappa_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r} \quad (1 \le r \le n).$$

We set $\sigma_0 = 1$. The r-th anisotropic mean curvature $H_{r;F}$ is defined by $H_{r;F} = \sigma_r / C_n^r$.

Using the characteristic polynomial of $S_F = -d\nu$, σ_r is defined by

$$\det(tI - S_F) = \sum_{r=0}^{n} (-1)^r \sigma_r t^{n-r}.$$

So, we have

$$\sigma_r = \frac{1}{r!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r} \delta^{j_1\cdots j_r}_{i_1\cdots i_r} h^{i_1}_{j_1} \cdots h^{i_r}_{j_r}, \tag{2.13}$$

where $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ is the usual generalized Kronecker symbol, i.e., $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ equals $+1$ (resp. -1) if $i_1 \cdots i_r$ are distinct and $(j_1 \cdots j_r)$ is an even (resp. odd) permutation of $(i_1 \cdots i_r)$ and in other cases it equals zero.

Definition 2.2. Let $f : M \to \mathbb{R}$ be a smooth function. We define the gradient $\operatorname{grad} f$ of the function f by

$$g_\nu(\operatorname{grad} f, X) = X(f), \tag{2.14}$$

where X is any smooth vector field on M.

Define f_i by $df = f_i \omega^i$, then

$$\operatorname{grad} f = g^{ij} f_j e_i. \tag{2.15}$$

We define

$$dV = |e_1, \cdots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \omega^n,$$

where $|e_1, \cdots, e_n, \nu|$ is the determinant of the matrix $(e_1, \cdots, e_n, \nu)$. Then dV is a volume form on M.

Definition 2.3. Let X be a smooth vector field on M. We define the divergence $\operatorname{div} X$ by $d\{i(X)dV\} = (\operatorname{div} X)dV$, where

$$(i(X)dV)(Y_1, \cdots, Y_{n-1}) \equiv dV(X, Y_1, \cdots, Y_{n-1}), \qquad \forall Y_1, \cdots, Y_{n-1} \in \mathscr{X}(M).$$

Lemma 2.4. *Let $X = X^i e_i$, then $\operatorname{div} X = X^i_i$, where*

$$dX^i + X^j \omega^i_j = X^i_j \omega^j.$$

Proof. By (2.6), (2.7), we get

$$d\,|e_1, \cdots, e_n, \nu| = \omega^i_i\, |e_1, \cdots, e_n, \nu|\,. \tag{2.16}$$

From the definition of $i(X)$, we have

$$i(X)dV = \sum_i (-1)^{i+1} X^i\, |e_1, \cdots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n.$$

So,

$$
\begin{aligned}
& d\{i(X)dV\} = \textstyle\sum_i (-1)^{i+1}(dX^i)\wedge |e_1,\cdots,e_n,\nu|\,\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n\\
+\ & \textstyle\sum_i (-1)^{i+1}X^i(d\,|e_1,\cdots,e_n,\nu|)\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n\\
+\ & \textstyle\sum_{j<i}(-1)^{i+j}X^i\,|e_1,\cdots,e_n,\nu|\,d\omega^j\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n\\
+\ & \textstyle\sum_{j>i}(-1)^{i+j+1}X^i\,|e_1,\cdots,e_n,\nu|\,d\omega^j\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^n\\
=\ & X^i_i dV.
\end{aligned}
$$

□

Remark 2.5. Recall $\nu = \phi\circ N = DF|_N + F(N)N$, so $dV = F(N)dA$, where dA is the area form of M induced by the standard Euclidean metric of $\mathbb{R}^{n+1}$. Thus $\int_M dV$ is just the anisotropic surface energy $\int_M F(N)dA$ which has been studied by many authors (see [5], [9], [12], [14], [21] etc.).

3. $L_{r;F}$ OPERATOR FOR HYPERSURFACES

We introduce an important operator P_r by

$$P_r = \sigma_r I - \sigma_{r-1}S_F + \cdots + (-1)^r S_F^r, \quad r = 0,\cdots,n,$$

then

$$P_0 = I,\ P_n = 0,\ P_r = \sigma_r I - P_{r-1}S_F.$$

Lemma 3.1. *The matrix of P_r is given by:*

$$(P_r)_i^j = \frac{1}{r!}\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} h^{i_1}_{j_1}\cdots h^{i_r}_{j_r}. \tag{3.1}$$

Proof. We prove Lemma 3.1 inductively. For $r=0$, it is easy to check that (3.1) is true.

We can check directly

$$\delta^{j_1\cdots j_q}_{i_1\cdots i_q} = \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{q-1}}_{i_1} & \delta^{j_q}_{i_1}\\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{q-1}}_{i_2} & \delta^{j_q}_{i_2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \delta^{j_1}_{i_{q-1}} & \delta^{j_2}_{i_{q-1}} & \cdots & \delta^{j_{q-1}}_{i_{q-1}} & \delta^{j_q}_{i_{q-1}}\\ \delta^{j_1}_{i_q} & \delta^{j_2}_{i_q} & \cdots & \delta^{j_{q-1}}_{i_q} & \delta^{j_q}_{i_q} \end{vmatrix} \tag{3.2}$$

Assume that (3.1) is true for $r = k$, we only need to show that it is also true for $r = k + 1$. For $r = k + 1$, Using (2.13) and (3.2), we have

$$\begin{array}{rcl} RHS \text{ of } (3.1) & = & \dfrac{1}{(k+1)!} \displaystyle\sum_{i_1,\cdots,i_{k+1};j_1,\cdots,j_{k+1}} \delta^{j_1\cdots j_{k+1}j}_{i_1\cdots i_{k+1}i} h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\ & = & \dfrac{1}{(k+1)!} \displaystyle\sum \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{k+1}}_{i_1} & \delta^{j}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{k+1}}_{i_2} & \delta^{j}_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{j_1}_{i_{k+1}} & \delta^{j_2}_{i_{k+1}} & \cdots & \delta^{j_{k+1}}_{i_{k+1}} & \delta^{j}_{i_{k+1}} \\ \delta^{j_1}_{i} & \delta^{j_2}_{i} & \cdots & \delta^{j_{k+1}}_{i} & \delta^{j}_{i} \end{vmatrix} h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\ & = & \dfrac{1}{(k+1)!} \displaystyle\sum (\delta^j_i \delta^{j_1\cdots j_{k+1}}_{i_1\cdots i_{k+1}} - \delta^{j_{k+1}}_i \delta^{j_1\cdots j_k j}_{i_1\cdots i_k i_{k+1}} + \cdots) h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\ & = & \sigma_{k+1}\delta^j_i - \dfrac{1}{(k+1)!} \displaystyle\sum \delta^{j_{k+1}}_i \delta^{j_1\cdots j_k j}_{i_1\cdots i_k i_{k+1}} h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} + \cdots \\ & = & \sigma_{k+1}\delta^j_i - \sum (P_k)^{i_{k+1}}_i h^j_{i_{k+1}} \\ & = & (P_{k+1})^j_i. \end{array}$$

□

Lemma 3.2. *For each r, we have*

(a) $(P_r)_i{}^j{}_j = 0$*;*

(b) $\mathrm{Trace}(P_r) = (n - r)\sigma_r$*;*

(c) $\mathrm{Trace}(P_r S_F) = (r + 1)\sigma_{r+1}$*;*

(d) $\mathrm{Trace}(P_r S_F^2) = \sigma_1\sigma_{r+1} - (r + 2)\sigma_{r+2}$*.*

Proof. (a). Noting (j, j_r) is skew symmetric in $\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}$ and (j, j_r) is symmetric in $h^{i_1}_{j_1}\cdots h_{j_r}{}^{i_r}{}_j$ (from (2.12)), we have

$$\sum_j (P_r)_i{}^j{}_j = \frac{1}{(r-1)!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r;j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} h^{i_1}_{j_1}\cdots h_{j_r}{}^{i_r}{}_j = 0.$$

(b). Using (3.1) and (2.13), we have

$$\begin{array}{rcl} \mathrm{Trace}(P_r S_F) & = & \sum_{ij} (P_r)^j_i h^i_j \\ & = & \frac{1}{r!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r;i,j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} h^{i_1}_{j_1}\cdots h^{i_r}_{j_r} h^i_j \\ & = & (r+1)\sigma_{r+1}. \end{array}$$

(c). Using (b) and the definition of P_r, we have

$$\mathrm{Trace}(P_r) = \mathrm{tr}(\sigma_r I) - \mathrm{tr}(P_{r-1} S_F) = n\sigma_r - r\sigma_r = (n - r)\sigma_r.$$

(d). Using (b) and the definition of P_{r+1}, we have

$$\mathrm{Trace}(P_r S_F^2) = \mathrm{Trace}(\sigma_{r+1} S_F) - \mathrm{Trace}(P_{r+1} S_F) = \sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}.$$

□

Remark 3.3. When $F = 1$, Lemma 3.2 was a well-known result (for example, see Barbosa-Colares [4], or Reilly [17]).

We define an operator $L_{r;F} : C^\infty(M) \to C^\infty(M)$ by

$$L_{r;F}(f) = \mathrm{div}(P_r \nabla f). \tag{3.3}$$

Proposition 3.4. *Let $x : M \to \mathbb{R}^{n+1}$ be a compact hypersurface in $\mathbb{R}^{n+1}$ with $H_{r;F} > 0$, then for $1 \le j \le r$,*

(1) each operator $L_{j;F}$ is elliptic;

(2) each j-mean curvature $H_{j;F}$ is positive.

Proof. See [4], Proposition 3.2, p. 280. □

4. Divergence Theorem and Minkowski integral formula

Lemma 4.1. $\mathrm{div}(P_r(x^T)) = c(r)(H_{r;F} + H_{r+1;F}\langle x, \nu\rangle_\nu)$.

Proof. We have

$$x = a^i e_i + \langle x, \nu\rangle_\nu \nu, \tag{4.1}$$

where $a^i = \langle x, e_j\rangle_\nu g^{ij}$. Differentiate it, we get

$$dx = (da^i)e_i + a^i de_i + d\langle x, \nu\rangle_\nu \nu + \langle x, \nu\rangle_\nu d\nu = \omega^i e_i.$$

Compare the coefficients of e_i, we have

$$da^i + a^j \omega_j^i - \langle x, \nu\rangle_\nu h_j^i \omega^j = \omega^i.$$

So,

$$a_j^i = \delta_j^i + s_j^i \langle x, \nu\rangle_\nu. \tag{4.2}$$

From Lemma 3.2, we compute

$$\begin{aligned}
\mathrm{div}(P_r(x^T)) &= ((P_r)_i^j a^i)_j \\
&= (P_r)_i^j a_j^i + (P_r)_i{}^j{}_j a^i \\
&= \mathrm{Trace}(P_r) + \mathrm{Trace}(P_r S_F)\langle x, \nu\rangle_\nu \\
&= c(r)(H_{r;F} + H_{r+1;F}\langle x, \nu\rangle_\nu).
\end{aligned}$$

□

Since M is compact without boundary, we have the following divergence theorem:

Lemma 4.2. (Divergence Theorem) $\int_M (\mathrm{div} X) dV = 0$.

Since $\mathrm{div}(fX) = f\mathrm{div}X + \langle \nabla f, X\rangle_\nu$, we have

Lemma 4.3. $\int_M (f\mathrm{div}X)dV + \int_M \langle \nabla f, X\rangle_\nu dV = 0$.

From Lemma 4.1 and the Divergence Theorem, we have the following Minkowski integral formula (see [9]):

Theorem 4.4. (Minkowski integral formula)

$$\int_M (H_r + H_{r+1}\langle x, \nu\rangle_\nu)dV = 0.$$

5. Proof of Theorem 1.2

We first prove the following lemma:

Lemma 5.1. *Let $X\colon M \to \mathbb{R}^{n+1}$ be a compact oriented hypersurface without boundary immersed into Euclidean space, and let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth function which satisfies the convexity condition (1.2). If $H_{r;F} > 0$ and*

$$\int_M x dV = 0, \tag{5.1}$$

then

$$\lambda_1^{L_{r;F}} \int_M \|x\|_\nu^2 dV \le \mu c(r) \int_M H_{r;F} dV. \tag{5.2}$$

Proof. From

$$\lambda_1^{L_{r;F}} = \inf_{\int_M f dV = 0} \frac{-\int_M f L_{r;F}(f) dV}{\int_M f^2 dV}, \tag{5.3}$$

writing $x = (x_1, \cdots, x_{n+1})$, $e_i = (e_{i1}, \cdots, e_{i,n+1})$ we have

$$\nabla x_\alpha = g^{ij} e_{i\alpha} e_j, \qquad \alpha = 1, \cdots, n+1.$$

Now, we choose $e_1, \cdots, e_n$ be orthonormal eigenvectors of S_F corresponding, respectively, to the eigenvalues $\kappa_1, \cdots, \kappa_n$. Represent by S_i the restriction of the transformation S_F to the subspace normal to e_i, and by $\sigma_r(S_i)$ the r-symmetric function associated to S_i. Then, we have

$$\begin{array}{rcl} \lambda_1^{L_{r;F}} \int_M x_\alpha^2 dV & \le & -\int_M x_\alpha L_{r;F}(x_\alpha) dV \\ & = & \int_M \langle P_r(\nabla x_\alpha), \nabla x_\alpha\rangle_\nu dV \\ & = & \int_M \sum_i \sigma_r(S_i) e_{i\alpha}^2 dV. \end{array} \tag{5.4}$$

Making summation over α from 1 to $n+1$, we get(by Proposition 3.4, $\sigma_r(S_i)$ is positive):

$$\begin{array}{rcl} \frac{1}{\Lambda}\lambda_1^{L_{r;F}}\int_M \|x\|_\nu^2 dV & \leq & \lambda_1^{L_{r;F}}\int_M \|x\|^2 dV \\ & \leq & \int_M \sum_i \sigma_r(S_i)\|e_i\|^2 dV \\ & \leq & \frac{1}{\lambda}\int_M \sum_i \sigma_r(S_i) dV \\ & = & \frac{c(r)}{\lambda}\int_M H_{r;F} dV. \end{array}$$

□

Proof of Theorem. Let $\int_M x dV = C$, constant vector in $\mathbb{R}^{n+1}$, then

$$\tilde{x} = x - \frac{1}{vol(M)}C$$

satisfies $\int_M \tilde{x} dV = 0$. Because the qualities of are the same for x and $\tilde{x}$, so holds for x is equivalent to that holds for $\tilde{x}$, so without loss of generality, we can assume that

$$\int_M x dV = 0.$$

Multiplying two sides of (5.2) by $\int_M H_{s;F}^2 dV$, we get

$$\lambda_1^{L_{r;F}}\int_M \|x\|_\nu^2 dV \cdot \int_M H_{s;F}^2 dV \leq \mu \cdot c(r)\int_M H_{r;F} dV \cdot \int_M H_{s;F}^2 dV \tag{5.5}$$

Using Schwartz inequality and Minkowski formula, we have

$$\begin{array}{rcl} \lambda_1^{L_{r;F}}\int_M \|x\|_\nu^2 dV \cdot \int_M H_{s;F}^2 dV & \geq & \lambda_1^{L_{r;F}}(\int_M \|x\|_\nu \cdot |H_{s;F}| dV)^2 \\ & \geq & \lambda_1^{L_{r;F}}(\int_M \langle x, H_{s;F}\nu\rangle_\nu dV)^2 \\ & = & \lambda_1^{L_{r;F}}(\int_M H_{s;F}\langle x, \nu\rangle_\nu dV)^2 \\ & = & \lambda_1^{L_{r;F}}(\int_M H_{s-1;F} dV)^2. \end{array}$$

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